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# Dynamic Boundary Stabilization of First Order Hyperbolic Systems

Felipe Castillo, Emmanuel Witrant, Christophe Prieur and Luc Dugard

**Abstract** In this chapter, we address the problem of the dynamic boundary stabilization of linear, quasilinear and LPV first-order hyperbolic systems. We provide sufficient conditions for the exponential stability for this class of infinite dimensional systems by means of Lyapunov based techniques and matrix inequalities. We develop an applicative example of a temperature boundary control in a Poiseuille flow using some of our main results and we present simulation results that illustrate the efficiency of our approach.

## 1 Introduction

Variable time-delays are often encountered in many industrial applications such as those where mass transport is present. This phenomenon makes the closed loop control much more challenging motivating the investigation of new strategies to effectively stabilize these systems. Among the potential applications, hydraulic networks [31], multiphase flow [24], road traffic networks [16], gas flow in pipelines [4] or flow regulation in deep pits [34] are of significant importance. Due to the convective nature of the mass transport, it is often modeled by means of balance laws which are described by hyperbolic partial differential equations (PDE) and typically used to express the fundamental dynamics of open conservative systems [35]. The interest in boundary control comes from the fact that measurements in distributed parameter systems are usually not available. It is more common for sensors and actuators to be located at the boundaries.

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Felipe Castillo, Emmanuel Witrant, Christophe Prieur and Luc Dugard  
GIPSA Lab, 11 rue des mathématiques, BP 46, 38402 Saint Martin d'Hères Cedex, France,  
e-mail: felipetillo@gmail.com, emmanuel.witrant@ujf-grenoble.fr,  
christophe.prieur@gipsa-lab.fr, luc.dugard@gipsa-lab.fr

Techniques based on Lyapunov functions are commonly used for the stability analysis of infinite dimensional dynamical systems, such as those described by strict hyperbolic partial differential equations. One of the main properties of this class of PDEs is the existence of the so-called Riemann transformation, which is a powerful tool for the proof of classical solutions, analysis and control, among other properties [2]. Control results for first-order hyperbolic systems do exist in the literature. For instance, sufficient conditions for controllability and observability of quasilinear hyperbolic systems have been obtained in [21]. The boundary control using backstepping designs and dissipative boundary conditions ([19, 18, 25, 32] and [12, 13, 5, 15, 27], respectively) are some of the most representative references for the boundary control of first-order hyperbolic systems. In [23], the boundary control problem is investigated using a frequency domain approach while [1] focuses in the disturbance rejection problem. In [29], a strict time-varying Lyapunov function that allows establishing the asymptotic stability of time-varying hyperbolic systems is formulated for time-invariant boundary conditions and in [28] the boundary control of switched linear hyperbolic systems is considered. Other important references on boundary control are found in [14, 20, 26, 30]. However, results in dynamic boundary stabilization of hyperbolic systems has been less explored and therefore we have proposed some strategies to address this aspects in our previous works [8, 7, 9].

In this chapter, we provide an extension of our previous results and general overview of the dynamic boundary stabilization of hyperbolic systems. We start with a brief description of the class of first-order hyperbolic systems considered in this work. In Section 3, we use Lyapunov-based techniques to establish sufficient conditions for exponential stability of linear, LPV and quasilinear hyperbolic systems with dynamics associated with their boundary conditions. A polytopic approach is developed for the LPV and quasilinear hyperbolic systems to guarantee the exponential stability inside a prescribed non-empty convex set. The main results are presented in three theorems using a matrix inequality framework. Finally in Section 4, a reduced physical model of an experimental setup for the regulation of the output temperature in a Poiseuille flow is developed. A temperature boundary control is designed using the main contributions of this work. Simulation results are presented to illustrate the effectiveness of the proposed control strategy.

**Notation.** By the expressions  $H \succeq 0$ ,  $H \preceq 0$ ,  $H \succ 0$  and  $H \prec 0$  we mean that the matrix  $H$  is positive semi-definite, negative semi-definite, positive definite and negative definite, respectively. The usual Euclidian norm in  $\mathbb{R}^n$  is denoted by  $|\cdot|$  and the associated matrix norm is denoted  $\|\cdot\|$ . Given  $g : [0, 1] \rightarrow \mathbb{R}^n$ , we define its  $L^2$ -norm (when is finite) as:

$$\|g\|_{L^2} = \sqrt{\int_0^1 |g(x)|^2 dx}$$

## 2 Linear and Quasi-Linear Hyperbolic Systems

Let  $n$  be a positive integer and  $\Theta$  be an open non-empty convex set of  $\mathbb{R}^n$ . Consider the following class of quasilinear hyperbolic systems of order  $n$ :

$$\partial_t \xi(x, t) + \Lambda(\xi) \partial_x \xi(x, t) = 0 \quad \forall x \in [0, 1], t \geq 0 \quad (1)$$

where  $\xi : [0, 1] \times [0, \infty) \rightarrow \mathbb{R}^n$  and  $\Lambda$  is a continuously differentiable diagonal matrix function  $\Lambda : \Theta \rightarrow \mathbb{R}^{n \times n}$  such that  $\Lambda(\xi) = \text{diag}(\lambda_1(\xi), \lambda_2(\xi), \dots, \lambda_n(\xi))$ .

**Remark 1.** Define the open non-empty convex set of  $\mathbb{R}^n$   $\Omega$  and consider the general class of quasilinear hyperbolic systems of order  $n$  defined as follows [22]:

$$\partial_t s(x, t) + F(s(x, t)) \partial_x s(x, t) = 0 \quad (2)$$

where  $s(x, t) \in \Omega$ , and  $F : \Omega \rightarrow \mathbb{R}^{n \times n}$  is a continuously differentiable function. If (2) is strictly hyperbolic (all eigenvalues of  $F(s)$  are different from zero and different between each other), then a local bijection  $\xi(s) \in \Theta \subset \mathbb{R}^n$  may exist, such that (2) can be transformed into the following system of coupled transport equations (it has been proved that for  $n \leq 2$ , this bijection always exists [11]):

$$\partial_t \xi_i(x, t) + \lambda_i(\xi(x, t)) \partial_x \xi_i(x, t) = 0, \quad i \in [1, \dots, n] \quad (3)$$

where  $\xi_i(x, t)$  are called the Riemann coordinates of (2), which are constant along the characteristic curves described by:

$$\frac{dx}{dt} = \lambda_i(\xi(x, t)) \quad (4)$$

and  $\xi = [\xi_1, \xi_2, \dots, \xi_n]^T$ . Using the transformation with Riemann coordinates, the system (2) can be expressed in the form of (1).

Let assume the following.

**Assumption 1.** The following inequalities hold for all  $\xi \in \Theta$ :

$$0 < \lambda_1(\xi) < \lambda_2(\xi) < \dots < \lambda_n(\xi) \quad (5)$$

If  $\Lambda(\xi) = \Lambda$ , then (1) is a linear hyperbolic system given by:

$$\partial_t \xi(x, t) + \Lambda \partial_x \xi(x, t) = 0 \quad (6)$$

We consider the following boundary controls for the quasilinear hyperbolic system (1):

$$\begin{aligned}\dot{X}_c &= A_c X_c(t) + B_c u_c(t) \\ Y_c(t) &= C_c X_c(t) + D_c u_c(t)\end{aligned}\tag{7}$$

with

$$Y_c(t) = \xi(0, t), \quad u_c = K\xi(1, t)\tag{8}$$

where  $X_c \in \mathbb{R}^{n_x}$ ,  $A_c \in \mathbb{R}^{n_x \times n_x}$ ,  $B_c \in \mathbb{R}^{n_x \times n}$ ,  $C_c \in \mathbb{R}^{n \times n_x}$ ,  $D_c \in \mathbb{R}^{n \times n}$ ,  $K \in \mathbb{R}^{n \times n}$ ,  $u \in \mathbb{R}^n$  and  $n_x \geq 1$ . Define the initial condition for (1) and (7) as:

$$\begin{cases} \xi(x, 0) = \xi^0(x), & \forall x \in [0, 1] \\ X_c(0) = X_c^0 \end{cases}\tag{9}$$

where  $\xi^0(x) \in L^2((0, 1); \mathbb{R}^n)$  and  $X_c^0 \in \mathbb{R}^{n_x}$ . It has been proved that there exist  $\delta_0 > 0$  and  $T > 0$  such that for every  $\xi^0 \in H^2((0, 1), \mathbb{R}^n)$  satisfying  $|\xi^0|_{H^2((0, 1), \mathbb{R}^n)} \leq \delta_0$  and the zero-order and one-order compatibility conditions, the Cauchy problem ((1), (7) and (9)) has a unique maximal classical solution satisfying (see e.g. [12] and [17], among other references):

$$|\xi(\cdot, t)|_{H^2((0, 1), \mathbb{R}^n)} \leq \delta_0 \quad \forall t \in [0, T]\tag{10}$$

Moreover, for linear hyperbolic systems (6),  $T = +\infty$  holds.

### 3 Dynamic Boundary Stabilization of Hyperbolic Systems

In this section, we give sufficient conditions for the dynamic boundary stabilization of linear, LPV and quasilinear hyperbolic systems (Sections 3.1, 3.2 and 3.3, respectively). More precisely, we address the problem of finding the boundary control gain  $K$  (according to (7)) such that (1) is exponentially stable.

#### 3.1 Stability of Linear Hyperbolic Systems with Dynamic Boundary Conditions

Sufficient conditions for the exponential stability of (6) with boundary conditions (7) and initial condition (9) are obtained with the following theorem:

**Theorem 1.** *[Stability analysis] Along with Assumption 1, assume that there exist two diagonal positive definite matrices  $P_1 \in \mathbb{R}^{n_x \times n_x}$  and  $P_2 \in \mathbb{R}^{n \times n}$  and a scalar  $\mu > 0$  such that the following matrix inequality is satisfied,*

$$M = \left[ \begin{array}{c|c} A_c^T P_1 + P_1 A_c + C_c^T \Lambda P_2 C_c & P_1 B_c K + C_c^T \Lambda P_2 D_c K \\ \hline + \mu \Lambda P_1 & \\ \hline K^T B_c^T P_1 + K^T D_c^T \Lambda P_2 C_c & K^T D_c^T \Lambda P_2 D_c K - e^{-\mu} \Lambda P_2 \end{array} \right] \preceq 0 \quad (11)$$

Then, there exist two constant values  $a > 0$  and  $b > 0$  such that, for all  $\xi^0 \in L^2((0, 1); \mathbb{R}^n)$  and  $X_c^0 \in \mathbb{R}^{n_x}$ , the solution of (6), (7) and (9) satisfies, for all  $t \geq 0$

$$\|X_c(t)\|^2 + \|\xi(t)\|_{L^2(0,1)}^2 \leq b e^{-at} \left( \|X_c^0\|^2 + \|\xi^0\|_{L^2(0,1)}^2 \right) \quad (12)$$

**Proof:** Given the diagonal positive definite matrices  $P_1$  and  $P_2$ , consider (as an extension of the Lyapunov function proposed in [13]) the quadratic strict Lyapunov function candidate defined for all continuously differentiable functions  $\xi : [0, 1] \rightarrow \Theta$  as:

$$V(\xi, X_c) = X_c^T P_1 X_c + \int_0^1 (\xi^T P_2 \xi) e^{-\mu x} dx \quad (13)$$

where  $\mu$  is a positive scalar. Note that (13) has some similarities with respect to the Lyapunov function proposed in [30] for boundary control with integral action. Computing the time derivative  $\dot{V}$  of  $V$  along the classical  $C^1$ -solutions of (6) with boundary conditions (7) and initial condition (9), yields to:

$$\dot{V} = \dot{X}_c^T P_1 X_c + X_c^T P_1 \dot{X}_c + \int_0^1 (\dot{\xi}^T P_2 \xi + \xi^T P_2 \dot{\xi}) e^{-\mu x} dx \quad (14)$$

After integration by parts, the following is obtained:

$$\begin{aligned} \dot{V} = & (X_c^T (A_c^T P_1 + P_1 A_c) X_c) + (\xi(1)^T K^T B_c^T P_1 X_c + X_c^T P_1 B_c K \xi(1)) \\ & - [e^{-\mu x} \xi^T \Lambda P_2 \xi] \Big|_0^1 - \mu \int_0^1 (\xi^T \Lambda P_2 \xi) e^{-\mu x} dx \end{aligned} \quad (15)$$

where  $\xi(1) = \xi(1, t)$ . The previous equation can be written using the boundary conditions (7) as:

$$\begin{aligned} \dot{V} = & (X_c^T (A_c^T P_1 + P_1 A_c) X_c) + (\xi(1)^T K^T B_c^T P_1 X_c + X_c^T P_1 B_c K \xi(1)) \\ & - e^{-\mu} \xi(1)^T \Lambda P_2 \xi(1) + X_c^T C_c^T \Lambda P_2 C_c X_c + X_c^T C_c^T \Lambda P_2 D_c K \xi(1) \\ & + \xi(1)^T K^T D_c^T \Lambda P_2 C_c X_c + \xi(1)^T K^T D_c^T \Lambda P_2 D_c K \xi(1) - \mu \int_0^1 (\xi^T \Lambda P_2 \xi) e^{-\mu x} dx \\ = & -\mu X_c^T \Lambda P_1 X_c - \mu \int_0^1 (\xi^T \Lambda P_2 \xi) e^{-\mu x} dx + \begin{bmatrix} X_c \\ \xi(1) \end{bmatrix}^T M \begin{bmatrix} X_c \\ \xi(1) \end{bmatrix} \end{aligned} \quad (16)$$

where the matrix  $M$  is defined as in (11). The matrix inequality  $M \preceq 0$  implies that the last term of (16) is always negative or zero. This gives the inequality:

$$\dot{V} \leq -\mu X_c^T \Lambda P_1 X_c - \mu \int_0^1 (\xi^T \Lambda P_2 \xi) e^{-\mu x} dx \quad (17)$$

From (5) it can be proved that there always exists a  $\rho > 0$  such that  $\Lambda - \rho I^{n \times n} \succeq 0$  (e.g.  $\rho$  could be the smallest eigenvalue of  $\Lambda$ ). Moreover, the diagonality of  $P_1$ ,  $P_2$  and  $\Lambda$  implies that:

$$\dot{V} \leq -\mu \rho V(\xi, X_c) \quad (18)$$

Therefore, the function (13) is a Lyapunov function for the hyperbolic system (6) with boundary conditions (7). Integrating the inequality (18) from 0 to  $t$  gives:

$$V(t) \leq V(0) e^{-\mu \rho t} \quad (19)$$

To obtain the final result (12), we bound the Lyapunov function as follows:

$$\begin{aligned} \min\{\lambda_{\min}(P_1), \lambda_{\min}(P_2)\}(\|X_c(t)\|^2 + \|\xi(t)\|_{L^2(0,1)}^2) &\leq \\ V(t) &\leq \max\{\lambda_{\max}(P_1), \lambda_{\max}(P_2)\}(\|X_c(t)\|^2 + \|\xi(t)\|_{L^2(0,1)}^2) \end{aligned} \quad (20)$$

where  $\lambda_{\min}$  and  $\lambda_{\max}$  are the minimum and maximum eigenvalues of the considered matrices, respectively. Then, using (19) together with (20) gives:

$$\|X_c(t)\|^2 + \|\xi(t)\|_{L^2(0,1)}^2 \leq \frac{\min\{\lambda_{\min}(P_1), \lambda_{\min}(P_2)\}}{\max\{\lambda_{\max}(P_1), \lambda_{\max}(P_2)\}} \left( \|X_c^0\|^2 + \|\xi^0\|_{L^2(0,1)}^2 \right) e^{-\mu \rho t} \quad (21)$$

which implies that  $a = \mu \rho$  and  $b = \min\{\lambda_{\min}(P_1), \lambda_{\min}(P_2)\} / \max\{\lambda_{\max}(P_1), \lambda_{\max}(P_2)\}$  in (12). ■

Note that the matrix inequality (11) considers, through the Lyapunov matrices  $P_1$  and  $P_2$ , the dynamic coupling between the system and its boundary conditions. Inequality (11) along with (18) implies that  $\mu$  is a tuning parameter of the controller design as it explicitly enables to set the convergence speed of the Lyapunov function. Another interesting convergence feature can be deduced from (18): a faster convergence is obtained for larger values of  $\rho$ . This implies that hyperbolic systems with high convective velocities converge faster, which is physically consistent. The following corollary gives a sufficient condition for the design of a stabilizing controller for the particular case where  $C_c$  is a diagonal matrix,  $D_c = 0$  and  $n = n_x$ .

**Corollary 1.** *[Design of a stabilizing controller] Along with Assumptions 1, if  $C_c$  is diagonal and  $D_c = 0$  and if there exist a diagonal positive definite matrix  $Q \in \mathbb{R}^{n \times n}$  and a scalar  $\mu > 0$  such that the following linear matrix inequality is satisfied,*

$$\begin{bmatrix} Q A_c^T + A_c Q + C_c \Lambda Q C_c + \mu \Lambda Q & B_c K Q \\ Q^T K^T B_c^T & -e^{-\mu} \Lambda Q \end{bmatrix} \preceq 0 \quad (22)$$

then there exist two values  $a > 0$  and  $b > 0$  such that, for all  $\xi^0(x) \in L^2((0, 1); \mathbb{R}^n)$  and  $X_c^0 \in \mathbb{R}^n$ , the solution of (6) with boundary conditions (7) and initial condition (9) satisfies (12) for all  $t \geq 0$ .

**Proof:** Given a diagonal positive definite matrix  $P$ , consider the quadratic strict Lyapunov function candidate defined for all continuously differentiable functions  $\xi : [0, 1] \rightarrow \Theta$  as:

$$V(\xi, X_c) = X_c^T P X_c + \int_0^1 (\xi^T P \xi) e^{-\mu x} dx \quad (23)$$

where  $\mu$  is a positive scalar. Performing the same procedure as the one presented in the proof of Theorem 1 and considering  $D_c = 0$ , the following equality is obtained

$$\dot{V} = -\mu X_c^T \Lambda P X_c - \mu \int_0^1 (\xi^T \Lambda P \xi) e^{-\mu x} dx + \begin{bmatrix} X_c \\ \xi(1) \end{bmatrix}^T H \begin{bmatrix} X_c \\ \xi(1) \end{bmatrix} \quad (24)$$

where

$$H = \begin{bmatrix} A_c^T P + P A_c + C_c \Lambda P C_c + \mu \Lambda P & P B_c K \\ K^T B_c^T P & -e^{-\mu} \Lambda P \end{bmatrix} \quad (25)$$

Note that (22) is equivalent to  $H \preceq 0$ . This is obtained by multiplying both sides of (25) by  $\text{diag}(P^{-1}, P^{-1})$ , commuting  $P^{-1}$  with  $C_c$  (both matrices being diagonal) and performing the variable transformations  $Q = P^{-1}$  and  $Y = KQ$ . Therefore, as  $H \preceq 0$  and  $\Lambda - \rho I^{n \times n} \succ 0$ , the function (23) is a Lyapunov function for the hyperbolic system (6) with boundary conditions (7). ■

This corollary is interesting because, for systems where  $C_c$  is diagonal and  $D_c = 0$ , it provides a constructive approach to obtain the boundary control gain  $K$  using convex optimization algorithms after determining a suitable value of  $\mu > 0$  (e.g. chosen to obtain a good performance versus robustness trade-off for the system considered).

### 3.2 Stability of Parameter-Varying Linear Hyperbolic Systems with Dynamic Boundary Conditions

Let  $Z_\varphi$  be a non empty convex set of  $\mathbb{R}^l$ . Consider the general class of first-order LPV hyperbolic systems of order  $n$  defined as follows:

$$\partial_t \xi(x, t) + \Lambda(\varphi) \partial_x \xi(x, t) = 0 \quad \forall x \in [0, 1], t \geq 0 \quad (26)$$

where  $\xi : [0, 1] \times [0, +\infty) \rightarrow \Theta$ ,  $\varphi$  is a varying parameter vector that takes values in the parameter space  $Z_\varphi$ ,  $\Lambda(\varphi) : Z_\varphi \rightarrow \mathbb{R}^{n \times n}$  is a diagonal and invertible matrix func-



tion (called the characteristic matrix) such that  $\Lambda(\varphi) = \text{diag}(\lambda_1(\varphi), \lambda_2(\varphi), \dots, \lambda_n(\varphi))$ .

**Assumption 2:** Assume that the following inequalities hold for all  $\varphi \in Z_\varphi$ :

$$0 < \lambda_1(\varphi) < \dots < \lambda_n(\varphi) \quad (27)$$

Consider the following dynamic boundary conditions for (26):

$$\begin{aligned} \dot{X}_c &= A_c(\varphi)X_c + B_c(\varphi)u \\ Y_c &= C_cX_c + D_cu \end{aligned} \quad (28)$$

with

$$Y_c = \xi(0, t), \quad u = K\xi(1, t) \quad (29)$$

where  $X_c \in \mathbb{R}^{n_x}$ ,  $A_c : Z_\varphi \rightarrow \mathbb{R}^{n_x \times n_x}$ ,  $B_c : Z_\varphi \rightarrow \mathbb{R}^{n_x \times n}$ ,  $C_c \in \mathbb{R}^{n \times n_x}$ ,  $D_c \in \mathbb{R}^{n \times n}$ ,  $K \in \mathbb{R}^{n \times n}$ ,  $u \in \mathbb{R}^n$  and  $n_x \geq 1$ . Define the initial condition for (26) as done in (9).

Let the polytope  $Z_\varphi$  be defined as follows:

$$Z_\varphi := \{[\varphi_1, \dots, \varphi_l]^T \in \mathbb{R}^l \mid \varphi_i \in [\underline{\varphi}_i, \overline{\varphi}_i], \forall i = 1, \dots, l\} \quad (30)$$

for given  $l \in \mathbb{N}^+$  and the parameter extremities  $\underline{\varphi}_i$ ,  $\overline{\varphi}_i$  (minimum and maximum, respectively). We thus consider that all the admissible values of the vector  $\varphi$  are constrained in a hyperrectangle in the parameter space  $Z_\varphi$ . Consider the polytopic linear representation of the parameter varying characteristic matrix for all  $\varphi \in Z_\varphi$  [3]:

$$\Lambda(\varphi) = \sum_{i=1}^{N_\varphi} \alpha_i(\varphi) \Lambda(w_i) \quad (31)$$

where  $w_i \in Z_\varphi$  are the  $N_\varphi = 2^l$  vertices of the polytope formed by all extremities ( $\overline{\varphi}_i$  and  $\underline{\varphi}_i$ ) of each varying parameter  $\varphi \in Z_\varphi$ ,  $\sum_{i=1}^{2^l} \alpha_i(\varphi) \Lambda(w_i) : Z_\varphi \rightarrow \mathbb{R}^{n \times n}$  and  $\alpha_i(\varphi)$  is a scheduling function  $\alpha_i : Z_\varphi \rightarrow [0, 1]$  defined as:

$$\alpha_i(\varphi) = \frac{\prod_{k=1}^l |\varphi_k - C(w_i)_k|}{\prod_{k=1}^l |\overline{\varphi}_k - \underline{\varphi}_k|} \quad (32)$$

where  $C(w_i)_k$  is the  $k$ th component of the vector  $C(w_i)$  defined as:

$$C(w_i)_k = \begin{cases} \overline{\varphi}_k, & \text{if } (w_i)_k = \underline{\varphi}_k \\ \underline{\varphi}_k, & \text{otherwise} \end{cases} \quad (33)$$

The scheduling functions  $\alpha_i$  have the following properties [3]:

$$\alpha_i(\varphi) \geq 0, \quad \sum_{i=1}^{N_\varphi} \alpha_i(\varphi) = 1 \quad (34)$$

The polytopic representation (31) can also be considered for the matrices  $A_c(\varphi)$  and  $B_c(\varphi)$  of the boundary conditions (28) using the same scheduling function (32). Define the polytopic linear representation (PLR) of the parameter varying hyperbolic system (26) with boundary conditions (28) as follows:

$$\begin{aligned} \partial_t \xi(x, t) + \sum_{i=1}^{N_\varphi} \alpha_i(\varphi) \Lambda(w_i) \partial_x \xi(x, t) &= 0 \\ \forall \varphi \in Z_\varphi, \quad \forall x \in [0, 1], \quad t &\geq 0 \end{aligned} \quad (35)$$

with boundary conditions

$$\begin{aligned} \dot{X}_c &= \sum_{i=1}^{N_\varphi} \alpha_i(\varphi) A_c(w_i) X_c + \sum_{i=1}^{N_\varphi} \alpha_i(\varphi) B_c(w_i) u \\ \xi(0, t) &= C_c X_c + D_c u \end{aligned} \quad (36)$$

Based on the PLR (35) - (36), the following theorem states a sufficient condition to ensure the exponential stability for system (26) with boundary conditions (28) and initial condition (9) for all  $\varphi \in Z_\varphi$ . This result is already proven in [10], but, since it will be useful for next sections, and the application considered in this chapter, let us recall it.

**Theorem 2.** *[Stability analysis] Along with Assumptions 2, assume that there exist two diagonal positive definite matrices  $P_1 \in \mathbb{R}^{n_x \times n_x}$  and  $P_2 \in \mathbb{R}^{n \times n}$  and a scalar  $\mu > 0$  such that the following matrix inequality is satisfied, for all  $i = 1, \dots, N_\varphi$ ,*

$$M_i = \left[ \begin{array}{c|c} A_c(w_i)^T P_1 + P_1 A_c(w_i) + C_c^T \Lambda(w_i) P_2 C_c & P_1 B_c(w_i) K + C_c^T \Lambda(w_i) P_2 D_c K \\ \hline + \mu \Lambda(w_i) P_1 & \\ \hline K^T B_c(w_i)^T P_1 + K^T D_c^T \Lambda(w_i) P_2 C_c & K^T D_c^T \Lambda(w_i) P_2 D_c K - e^{-\mu} \Lambda(w_i) P_2 \end{array} \right] \preceq 0 \quad (37)$$

Then there exist two constant values  $a > 0$  and  $b > 0$  such that, for all  $\xi^0 \in L^2((0, 1); \mathbb{R}^n)$  and  $X_c^0 \in \mathbb{R}^{n_x}$ , the solution of (26), (28) and (9) satisfies (12), for all  $t \geq 0$ .

**Proof:** Consider once again the Lyapunov function candidate (13). Computing the time derivative  $\dot{V}$  of  $V$  along the classical  $C^1$ -solutions of (26) with boundary conditions (28) and initial conditions (9), gives the following:

$$\dot{V} = \dot{X}_c^T P_1 X_c + X_c^T P_1 \dot{X}_c + \int_0^1 \left( \dot{\xi}^T P_2 \xi + \xi^T P_2 \dot{\xi} \right) e^{-\mu x} dx \quad (38)$$

After integration by parts and taking into account (26) and (28), the following is obtained:

$$\begin{aligned} \dot{V} = \sum_{i=1}^{2l} \alpha_i(\varphi) & \left[ \left( X_c^T (A_c(w_i)^T P_1 + P_1 A_c(w_i)) X_c \right) + \left( \xi(1)^T K^T B_c(w_i)^T P_1 X_c \right. \right. \\ & \left. \left. + X_c^T P_1 B_c(w_i) K \xi(1) \right) - \left[ e^{-\mu x} \xi^T \Lambda(w_i) P_2 \xi \right] \Big|_0^1 - \mu \int_0^1 \left( \xi^T \Lambda(w_i) P_2 \xi \right) e^{-\mu x} dx \right] \end{aligned} \quad (39)$$

The previous equation can be written using the boundary conditions (28) as follows:

$$\begin{aligned} \dot{V} = \sum_{i=1}^{2l} \alpha_i(\varphi) & \left[ \left( X_c^T (A_c(w_i)^T P_1 + P_1 A_c(w_i)) X_c \right) + \left( \xi(1)^T K^T B_c(w_i)^T P_1 X_c \right. \right. \\ & + X_c^T P_1 B_c(w_i) K \xi(1) \Big) - e^{-\mu} \xi(1)^T \Lambda(w_i) P_2 \xi(1) + X_c^T C_c^T \Lambda(w_i) P_2 C_c X_c \\ & + X_c^T C_c^T \Lambda(w_i) P_2 D_c K \xi(1) + \xi(1)^T K^T D_c^T \Lambda(w_i) P_2 C_c X_c \\ & + \xi(1)^T K^T D_c^T \Lambda(w_i) P_2 D_c K \xi(1) - \mu \int_0^1 \left( \xi^T \Lambda(w_i) P_2 \xi \right) e^{-\mu x} dx \Big] \\ & = \sum_{i=1}^{2l} \alpha_i(\varphi) \left[ -\mu X_c^T \Lambda(w_i) P_1 X_c - \mu \int_0^1 \left( \xi^T \Lambda(w_i) P_2 \xi \right) e^{-\mu x} dx \right. \\ & \left. + \begin{bmatrix} X_c \\ \xi(1) \end{bmatrix}^T M_i \begin{bmatrix} X_c \\ \xi(1) \end{bmatrix} \right] \end{aligned} \quad (40)$$

where the matrix  $M_i$  is defined as in (37). The definition  $\alpha_i \geq 0$  and the matrix inequality  $M_i \preceq 0$  from (37) imply that the last term of (40) is always negative or zero. This gives the following inequality:

$$\dot{V} \leq \sum_{i=1}^{2l} \alpha_i(\varphi) \left[ -\mu X_c^T \Lambda(w_i) P_1 X_c - \mu \int_0^1 \left( \xi^T \Lambda(w_i) P_2 \xi \right) e^{-\mu x} dx \right] \quad (41)$$

From (27), it can be proved that there always exists a  $\rho > 0$  such that  $\Lambda(\varphi) - \rho I^{n \times n} \succeq 0$  (e.g.  $\rho$  could be the smallest eigenvalue of  $\Lambda(\varphi)$  over  $Z_\varphi$ ). Moreover, the diagonality of  $P_1$ ,  $P_2$  and  $\Lambda$  implies that:

$$\dot{V} \leq -\mu \rho V(\xi, X_c) \quad (42)$$

Therefore, the function (13) is a Lyapunov function for the hyperbolic system (26) with boundary conditions (28). ■

Note that Theorem 2 allows verifying whether the boundary control gain  $K$  stabilizes the hyperbolic system (26) with boundary conditions (28) for all varying parameters that belong to the convex set  $Z_\varphi$ .

As done for linear hyperbolic systems, the following corollary gives sufficient conditions for the design of a stabilizing controller for the particular case where  $C_c$  is a diagonal matrix,  $D_c = 0$  and  $n = n_x$ .

**Corollary 2.** *[Design of a stabilizing controller] Along with Assumptions 2, if  $C_c$  is diagonal and  $D_c = 0$  and if there exist a diagonal positive definite matrix  $Q \in \mathbb{R}^{n \times n}$  and a scalar  $\mu > 0$  such that the following linear matrix inequality is satisfied, for all  $i \in 1, \dots, N_\phi$ ,*

$$\begin{bmatrix} QA_c(w_i)^T + A_c(w_i)Q + C_c\Lambda(w_i)QC_c + \mu\Lambda(w_i)Q & B_c(w_i)Y \\ Y^TB_c(w_i)^T & -e^{-\mu}\Lambda(w_i)Q \end{bmatrix} \preceq 0 \quad (43)$$

where  $Y = KQ$ , then there exist two values  $a > 0$  and  $b > 0$  such that, for all  $\xi^0(x) \in L^2((0, 1); \mathbb{R}^n)$  and  $X_c^0 \in \mathbb{R}^n$ , the solution of (26) with boundary conditions (28) and initial condition (9) satisfies (12) for all  $t \geq 0$ .

**Proof:** Consider once again the Lyapunov function candidate (23). Performing the same procedure as the one presented in the proof of Theorem 2 and considering  $D_c = 0$ , the following equality is obtained

$$\begin{aligned} \dot{V} = & \sum_{i=1}^{2^l} \alpha_i(\varphi) \left[ -\mu X_c^T \Lambda(w_i) P X_c - \mu \int_0^1 (\xi^T \Lambda(w_i) P \xi) e^{-\mu x} dx \right. \\ & \left. + \begin{bmatrix} X_c \\ \xi(1) \end{bmatrix}^T H_i \begin{bmatrix} X_c \\ \xi(1) \end{bmatrix} \right] \end{aligned} \quad (44)$$

where

$$H_i = \begin{bmatrix} A_c(w_i)^T P + PA_c(w_i) + C_c\Lambda(w_i)PC_c + \mu\Lambda(w_i)P & PB_c(w_i)K \\ K^TB_c(w_i)^TP & -e^{-\mu}\Lambda(w_i)P \end{bmatrix} \quad (45)$$

Note that (22) is equivalent to  $H_i \preceq 0$ . This is obtained by multiplying both sides of (45) by  $\text{diag}(P^{-1}, P^{-1})$ , commuting  $P^{-1}$  with  $C_c$  (both matrices being diagonal) and performing the variable transformations  $Q = P^{-1}$  and  $Y = KQ$ . Therefore, as  $H_i \preceq 0$  and  $\Lambda(\varphi) - \rho I^{n \times n} \succ 0$ , the function (23) is a Lyapunov function for the hyperbolic system (26) with boundary conditions (28). ■

### 3.3 Stability of Quasi-Linear Hyperbolic Systems with Dynamic Boundary Conditions

Some sufficient conditions have been found in [12] to exponentially stabilize one-dimensional quasilinear hyperbolic systems with static boundary conditions on a bounded interval by means of Lyapunov techniques.

To represent the variation of the characteristic matrix with respect to the state, let us define the convex set  $Z_\xi$  as follows:

$$Z_\xi := \{[\xi_1, \dots, \xi_n]^T \in \mathbb{R}^n \mid \xi_i \in [\bar{\xi}_i, \underline{\xi}_i], \forall i = 1, \dots, n\} \quad (46)$$

where  $\bar{\xi}_i$  and  $\underline{\xi}_i$  are some maximal and minimal values for  $\xi_i$ , respectively. In other words, the matrix  $\Lambda(\xi)$  belongs to the matrix variation domain [6]:

$$D_\xi = \left\{ \Lambda : \Lambda = \sum_{i=1}^{2^n} \beta_i \Lambda(v_i), \beta_i \geq 0, \sum_{i=1}^{2^n} \beta_i = 1 \right\} \quad (47)$$

where  $v_i \in Z_\xi$  are the  $N_\xi = 2^n$  vertices of the polytope formed by all the state extremities  $\bar{\xi}_i$  and  $\underline{\xi}_i$ . From this formulation we propose the following theorem, which gives sufficient conditions for the exponential stability of (1) with boundary conditions (7) and initial conditions (9) over the characteristic matrix variation domain  $D_\xi$ .

**Theorem 3.** *[Stability analysis] Along with Assumptions 1, assume that there exist two diagonal positive definite matrices  $P_1 \in \mathbb{R}^{n_x \times n_x}$  and  $P_2 \in \mathbb{R}^{n \times n}$  and a scalar  $\mu > 0$  such that the following matrix inequality is satisfied, for all  $i = 1, \dots, N_\xi$ ,*

$$\begin{bmatrix} A_c^T P_1 + P_1 A_c + C_c^T \Lambda(v_i) P_2 C_c & P_1 B_c K + C_c^T \Lambda(v_i) P_2 D_c K \\ + \mu \Lambda(v_i) P_1 & \\ \hline K^T B_c^T P_1 + K^T D_c^T \Lambda(v_i) P_2 C_c & K^T D_c^T \Lambda(v_i) P_2 D_c K - e^{-\mu} \Lambda(v_i) P_2 \end{bmatrix} \preceq 0 \quad (48)$$

Then there exist two constant values  $a > 0$  and  $b > 0$  such that, for all  $\xi^0 \in Z_\xi$  and  $X_c^0 \in \mathbb{R}^{n_x}$ , the solution of (1), (7) and (9) satisfies (12), for all  $t \geq 0$ .

**Proof:** The proof of this theorem is very similar to the one developed for Theorem 2. The main difference is that in this case, the stability is proved over the convex set of the state variation  $Z_\xi$  instead of the parameter space  $Z_\varphi$ . ■

Similarly to the Corollaries 1 and 2, the following corollary can be formulated for quasilinear hyperbolic systems:

**Corollary 3.** *[Design of a stabilizing controller] Along with Assumption 1, if  $C_c$  is diagonal and  $D_c = 0$  and if there exist a diagonal positive definite matrix  $Q \in \mathbb{R}^{n \times n}$  and a scalar  $\mu > 0$  such that the following linear matrix inequality is satisfied, for all  $i \in 1, \dots, N_\xi$ ,*

$$\begin{bmatrix} Q A_c^T + A_c Q + C_c \Lambda(v_i) Q C_c + \mu \Lambda(v_i) Q & B_c Y \\ Y^T B_c^T & -e^{-\mu} \Lambda(v_i) Q \end{bmatrix} \preceq 0 \quad (49)$$

where  $Y = KQ$ , then there exist two values  $a > 0$  and  $b > 0$  such that, for all  $\xi^0(x) \in Z_\xi$  and  $X_c^0 \in \mathbb{R}^{n_x}$ , the solution of (1) with boundary conditions (7) and initial condition (9) satisfies (12), for all  $t \geq 0$ .

This corollary completes our results on dynamic boundary stabilization of hyperbolic systems.

## 4 Boundary Temperature Control Example

To further investigate the phenomenon of fluid transport in a Poiseuille flow with dynamics at the boundary conditions, an experimental setup has been designed to test and validate advanced control strategies. Figure 1 shows the schematic of the proposed device.

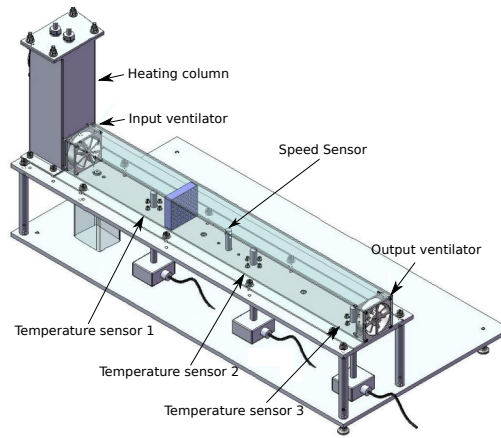


Fig. 1: Schematic of the experimental setup

This device mainly consists of a heating column encasing a resistor, a tube, two ventilators, a gas velocity meter and distributed temperature sensors. The control problem is to regulate the outlet temperature of the tube by driving the power dissipated on the heating resistor at different air flow speeds (exogenous inputs produced by fans) through the tube. Only the outlet temperature and the flow speed will be considered as measurements for the closed loop boundary control strategy.

The modeling of the experimental setup presented in Figure 1 is done by considering two subsystems: the heating column and the tube. A zero-dimensional model (0-D) (control volume approach with heat exchanges coming from the heating resistor) represents the heating column. For the tube, a one dimensional (1-D) transport model is used to describe the gas density variations in the tube.

### 4.1 Heating Column Model

Figure 2 presents the schematic of the 0-D control volume approach considered for the heating column.

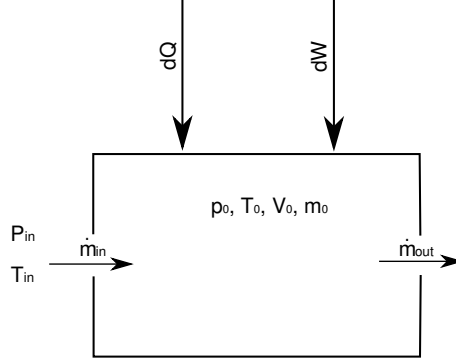


Fig. 2: Schematic of the control volume approach

Consider the internal energy of a perfect gas:

$$U_0 = C_v m_0 T_0 \quad (50)$$

where  $U_0$  is the gas internal energy,  $T_0$  the gas temperature,  $m_0$  the mass inside the column and  $C_v$  the specific heat of the gas for constant volume. The time derivative of (50) is:

$$\dot{U}_0 = C_v m_0 \dot{T}_0 + C_v T_0 \dot{m}_0 \quad (51)$$

Using the first law of thermodynamics, the dynamics of the internal energy of the gas inside the column can also be given by:

$$\dot{U}_0 = \sum h_i \dot{m}_i + dQ + dW \quad (52)$$

where  $h_i$  is the specific enthalpy getting in and out of the volume with a mass flow rate  $\dot{m}_i$ ,  $dQ$  quantifies the heat exchanges and  $dW$  is the work done by the gas. In the case of the heating column, there are two flows interacting with the volume characterized by the input mass flow rate  $\dot{m}_{in}$  and the output mass flow rate  $\dot{m}_{out}$ . As the gas does not perform any work, then  $dW = 0$ . In order to write (52) in terms of temperature, the specific enthalpy of a gas, defined by  $h = C_p T$ , where  $C_p$  is the specific heat constant at constant pressure, is used. Therefore, (52) can be expressed as:

$$\dot{U}_0 = C_p T_{in} \dot{m}_{in} - C_p T_0 \dot{m}_{out} + dQ \quad (53)$$

where  $T_{in}$  is the heating column input temperature. To simplify the model, consider the following set of two hypotheses:

**Hypotheses 1:**

- The pressure dynamics is much faster than the temperature dynamics, which allows considering a quasi-steady behavior of the mass and pressure;
- $p_0 \approx p_{in}$ , where  $p_{in}$  is the input pressure;

Hypotheses 1 allow writing (51) and (53), respectively as:

$$\dot{U}_0 = C_v m_0 \dot{T}_0 \quad (54)$$

$$\dot{U}_0 = C_p \dot{m}_{in} (T_{in} - T_0) + dQ \quad (55)$$

To simplify, overall in the 1-D model (see next sub-section), the temperature dynamics can be expressed in terms of the gas density by introducing the following change of variable (perfect gases law):

$$\rho_0 = \frac{p_{in}}{RT_0} \quad (56)$$

Taking the time derivative of (56) yields

$$\dot{T}_0 = -\frac{R}{p_{in}} T_0^2 \dot{\rho}_0 \quad (57)$$

Equalizing (54) and (55) and using (56) - (57) along with the perfect gases law to replace the mass inside the control volume  $m_0$  in terms of the pressure and the specific gas constant  $R$ , the following is obtained:

$$\dot{\rho}_0 = -\frac{R\gamma T_{in} \dot{m}_{in}}{p_{in} V_0} \rho_0 - \frac{R}{p_{in} V_0 C_v} \rho_0 dQ + \frac{\gamma \dot{m}_{in}}{V_0} \quad (58)$$

where  $\rho_0 = m_0/V_0$  is the density inside the heating column,  $V_0$  is the column volume and  $\gamma = \frac{C_p}{C_v}$ .



## 4.2 Tube Model

To model the dynamics of the fluid inside the tube, the one-dimensional Euler equations are considered for a perfect gas and a constant tube cross section. These equations can be written in terms of the primitive variables (density  $\rho$ , particle speed  $u$  and pressure  $p$ ) as follows (see [33]):

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{A}(\mathbf{V}) \frac{\partial \mathbf{V}}{\partial x} + \mathbf{C}(\mathbf{V}) = 0 \quad (59)$$

$$\mathbf{V} = \begin{bmatrix} \rho \\ u \\ p \end{bmatrix}; \mathbf{A} = \begin{bmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & a^2 \rho & u \end{bmatrix}; \mathbf{C} = \begin{bmatrix} 0 \\ G \\ (\gamma - 1)\rho(q + uG) \end{bmatrix}$$

where  $a = \sqrt{\frac{\gamma p}{\rho}}$  is the speed of sound,  $G$  is a term associated with the friction losses and  $q$  is a term associated with the wall heat exchanges. In order to simplify (59), the following set of hypotheses are considered.

### Hypotheses 2:

- the propagation speed of the entropy wave (average energy and mass) inside the tube is much slower than the speed of the sound  $u \ll a$ ;
- the pressure inside the tube is considered constant (equal to the atmospheric pressure) because the pressure differential introduced by the fans is very small;
- the heat exchanges and the friction in the tube are neglected:  $q = 0$  and  $G = 0$ .

Hypotheses 2 imply  $\frac{\partial u(x,t)}{\partial x} \approx 0$  and  $\frac{\partial p(x,t)}{\partial x} = 0$ . This reduces system (59) to the following convection equation:

$$\frac{\partial \rho(x,t)}{\partial t} + u(t) \frac{\partial \rho(x,t)}{\partial x} = 0 \quad (60)$$

where  $u(t)$  is the time-varying convection parameter of (60). The gas speed  $u(t)$  in the tube is measured. Using Hypotheses 2 allows expressing the input mass flow rate as:

$$\dot{m}_{in} = u(t)\rho(0,t)A_t \quad (61)$$

where  $A_t$  is the tube cross section area. With (61), the boundary conditions of (60) can be expressed as:

$$\begin{aligned}\dot{\rho}_0 &= -\frac{R\gamma T_{in}u(t)\rho(0,t)A_t}{p_{in}V_0}\rho_0 - \frac{R}{p_{in}V_0C_v}\rho_0 dQ + \frac{\gamma u(t)\rho(0,t)A_t}{V_0}; \\ \rho(0,t) &= \rho_0\end{aligned}\quad (62)$$

### 4.3 Output Temperature Boundary Control

The output temperature boundary control is designed for (60) with boundary conditions (62). Define the density error as:

$$\xi = \rho - \rho_{ref} \quad (63)$$

where  $\rho_{ref}$  is the desired output density. It is easy to show that system (60) with boundary conditions (62) can be expressed in an LPV form as follows:

$$\frac{\partial \xi(x,t)}{\partial t} + \Lambda(\varphi) \frac{\partial \xi(x,t)}{\partial x} = 0 \quad (64)$$

with boundary conditions:

$$\dot{\xi}_0 = A_c(\varphi)\xi_0 + B_c(\varphi)dQ + E_c(\varphi) \quad (65)$$

and with the varying parameters defined as follows:

$$\varphi_1 = u(t), \quad \varphi_2 = \rho_0, \quad \varphi_3 = \rho_0 u(t), \quad (66)$$

for suitable functions  $A_c(\varphi)$ ,  $B_c(\varphi)$  and  $\Lambda(\varphi)$ .

Only  $\varphi_1$  is measured.  $\varphi_2$  and  $\varphi_3$  are considered as uncertain parameters as no temperature measurement is taken inside the heating column. Strictly speaking, system (65) is quasi-LPV because one of the parameters is a state. However, as  $\rho_0$  can be easily bounded from the knowledge of the operating conditions of the experimental setup, then the system can be considered as an LPV one. Define the control input as:

$$dQ = F(\varphi, \rho_{ref}) + K\xi(L,t) \quad (67)$$

where  $F(\varphi, \rho_{ref})$  is defined as:

$$\begin{aligned}F(\varphi, \rho_{ref}) &= -B_c(\varphi)^{-1}E_c(\varphi) \\ &= C_v\gamma u(t)A_t \left( \frac{p_{in}}{R} - T_{in}\rho_{ref} \right)\end{aligned}\quad (68)$$

This yields to the system (60) with boundary conditions (62), which corresponds to the system considered in Theorem 2. Note that  $dQ$  is independent of the uncertain parameters  $\varphi_2$  and  $\varphi_3$ , which is crucial for the boundary control implementation.

Define the convex subset  $Z_\varphi$  in order to operate the experimental setup between the temperatures of 290 K and 323 K and a flow speed between 0.63 m/s and 3.82 m/s:

$$Z_\varphi := \{[\varphi_1, \varphi_2, \varphi_3]^T \in \mathbb{R}^3, \varphi_1 \in [0.63, 3.82], \varphi_2 \in [0.968, 1.08], \varphi_3 \in [0.61, 4.12]\} \quad (69)$$

In order to design the boundary control for system (60) with boundary conditions (62), consider the control architecture presented in Figure 3.

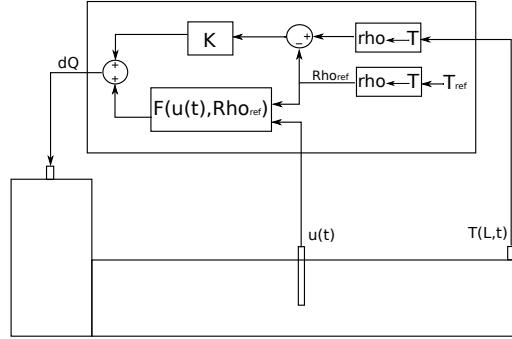


Fig. 3: Flow Tube Control Architecture

#### 4.4 Simulation results

Consider the following system parameters:  $p_{in} = 1 \times 10^5$  Pa,  $T_{in} = 300$  K,  $V_0 = 4 \times 10^{-3}$  m<sup>3</sup>,  $A_t = 6.4 \times 10^{-3}$  m<sup>2</sup>, and  $L = 1.5$  m where  $L$  is the tube's length. Applying Theorem 2 leads to the following control gain and Lyapunov function parameter  $P$ :

$$K = -654, \quad P = 1 \quad (70)$$

which ensure that the equilibrium  $\xi = 0$  is exponentially stable  $\forall \varphi \in Z_\varphi$ . In order to illustrate the effectiveness of the proposed boundary control strategy, some simulation results of system (60) with boundary conditions (62) and the boundary control (70) are presented for different flow speeds. The results obtained are presented in Figures 4 and 5. A change of temperature reference from 300K to 320K (which can be transformed into a density reference using the perfect gases law) is introduced at 1s.

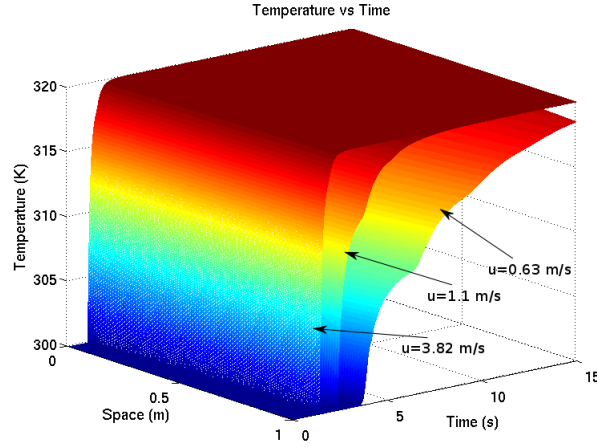


Fig. 4: Output temperature boundary control results for 3 different flow speeds

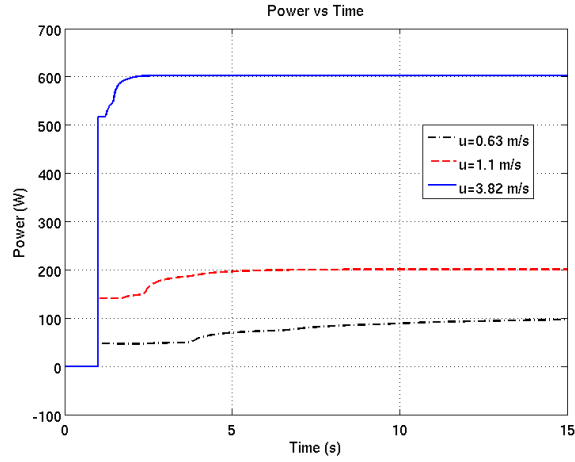


Fig. 5: Control Input for 3 different flow speeds

As depicted in Figure 4, the system effectively follows the change of reference for the different flow speeds. The faster the flow speed, the faster the convergence, as the fluid transport time is smaller as remarked from the proof of Theorem 1. Figure 5 shows the respective control inputs obtained for the simulation results. It appears that the power dissipated by the heating resistor has to be greater as the flow speed increases. This is due to the fact that in this case, the gas residence time inside the heating column is smaller and the amount of energy absorbed by the gas is less important.

## 5 Chapter Summary

This chapter focused on the stabilization of linear, LPV and quasilinear first-order hyperbolic systems with dynamic boundary conditions and their application to temperature boundary control in a Poiseuille flow. We found sufficient conditions for boundary control design for linear, LPV and quasilinear strict hyperbolic systems with  $n$  rightward convecting PDEs with dynamic boundary conditions. An extension of the strict Lyapunov function proposed in [13] was used to demonstrate the exponential stability of this class of infinite dimensional systems. In Theorem 1, we presented the sufficient conditions for the boundary control design for linear hyperbolic systems with dynamic boundary control, for all  $\xi^0 : [0, 1] \rightarrow \Theta$ . In Theorem 2, a polytopic formulation was considered to state sufficient conditions for boundary control design for LPV hyperbolic systems with LPV dynamics at the boundary conditions over a convex set  $Z_\phi$ . Then, in Theorem 3, by representing the nonlinear characteristic matrix  $\Lambda(\xi)$  in a polytopic representation, sufficient conditions for exponential stability were found for all  $\xi^0 \in Z_\xi$ . Finally, an application of the main results of this chapter are illustrated by developing a temperature boundary control for an experimental setup specially designed to study the mass transport. Simulation results have shown the effectiveness of the proposed dynamic boundary control techniques.

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